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Numerical Analysis of Second Order Differential Equations Via Haar Wavelets

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ABSTRACT

The efficiency, simplicity, and excellent approximation capabilities of wavelet-based numerical techniques in solving differential equations in science and engineering have made them a hot topic in recent years. Piecewise constant basis functions and ease of implementation define the Haar wavelet technique as one of the oldest and most basic approaches. This research finds a numerical solution to an ordinary differential equation of second order by applying the Haar wavelet algorithm. The computational complexity is greatly reduced by reducing the controlling differential equation to a set of algebraic equations using the operational matrix of integration. In order to compare analytical and numerical findings side by side, the approach is applied to a test issue that already has an exact solution. At several collocation locations, the numerical results are in good agreement with the actual solution, with modest absolute errors. The findings validate the accuracy and efficacy of the Haar wavelet approach as a numerical tool; furthermore, it is applicable to many boundary and starting value issues, and its performance may be improved by adding more collocation points.

Keywords: *Haar Wavelets, Operational Matrix, Differential Equation, Numerical Method.*

I. Introduction

Heat transfer, fluid movement, population dynamics, electromagnetic fields, and financial systems are just a few examples of the many physical, biological, and technical processes that rely on differential equations for their modelling. Ordinary and partial differential equations regulate many practical issues, and finding their accurate analytical solutions is a formidable challenge, if not an impossibility. So, for the purpose of estimating solutions to these equations, numerical and semi-analytical approaches are now essential. Numerous numerical techniques, including variational approaches, spectral methods, finite element methods, and finite difference methods, have been developed and used extensively throughout the years. On the other hand, computational complexity, convergence speed, and dealing with non-smooth solutions are common issues that these conventional approaches face. Here, wavelet-based approaches have shown to be strong and versatile substitutes for traditional methods of solving differential equations.

The mathematical foundation for describing functions at many levels of resolution was established by wavelet theory in the latter part of the twentieth century. Wavelets are able to capture both global trends and local changes in a function, unlike standard Fourier techniques that depend on global basis functions. This is because wavelets employ localised basis functions. For differential equations with abrupt gradients, discontinuities, or localised events, wavelets' localisation property is ideal. Haar wavelets are very appealing for numerical applications since they are the most computationally efficient and simplest version of the many wavelet families.

Defined over a limited interval, the piecewise constant function known as the Haar wavelet was first proposed by Alfréd Haar in 1910. A full orthogonal basis in the space of square-integrable functions is formed by the Haar wavelet, which is rather simple, despite this. To approximatively solve differential equations involving unknown functions and their derivatives, the Haar wavelet algorithm takes advantage of this orthogonality and compact support. The Haar wavelet excels in dealing with issues involving piecewise-defined coefficients, non-smooth solutions, or sudden changes because of its step-like structure. Differentiating Haar wavelets from higher-order wavelets—which could have more intricate calculations and smoother solution spaces—are these features.

The unknown function is usually extended as a finite sequence of Haar wavelet basis functions when solving differential equations using the Haar wavelet technique. By transforming differential equations into systems of algebraic equations, operational matrices of integration express the function's derivatives. Because solving algebraic systems is typically easier and faster than working with differential operators directly, this transformation greatly simplifies the computing process. In particular, the approach works for integro-differential equations, fractional differential equations, ordinary and nonlinear differential equations, partial differential equations, and so on.

The adaptability and resilience of the Haar wavelet approach have led to its rising profile in recent years within the realms of science and engineering. Equations involving heat and diffusion, waves, reaction-diffusion systems, models of fluid dynamics, and control systems have all found use for it. Signal processing, data analysis, and picture compression are just a few of the multidisciplinary fields that have discovered uses for the technology. It is clear from these fruitful uses that Haar wavelets are a powerful computing tool for solving differential equations.

II. Review of Literature

Shiralashetti, Siddu & E., Harishkumar (2020) below we give the numerical solution of the system of ordinary differential equations using the Haar wavelet technique. Because of its ease of use and effectiveness in numerical approximations, the Haar wavelet basis is being considered as a potential solution to the problem. The numerical solution of a system of equations using the Haar wavelet algorithm is discussed, and its results are compared to the precise solution. In addition, numerical data are provided to show that the Haar wavelet approach is legitimate and useful.

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Shah, Firdous & Abbas, Reyadh (2015) To numerically solve fractional order differential equations, we provide a novel operational matrix approach of fractional order integration based on Haar wavelets in this study. We begin by outlining the features of Haar wavelets. In order to simplify the system of fractional order differential equations to an algebraic equation system that can be solved numerically using Newton's technique, the characteristics of Haar wavelets are employed. In addition, the inverse of the Haar matrices is not necessary and the block pulse functions that are examined in the open literature are not used to derive the suggested technique. Included numerical examples show the method's validity and its usefulness.

Gopalakrishnan, Hariharan & Kannan, K. (2013) an increasing amount of focus in engineering research has been on investigating different wavelet approaches due to its capacity to analyse a wide range of dynamic phenomena through waves. Wavelets are useful tools for creating accurate mathematical models of scientific phenomena, which are often represented by linear or nonlinear differential equations. Their applications range from "offering good solution to differential equations" to capturing nonlinearity in data distribution. According to the review, the Haar wavelet technique (HWM) is a strong and efficient tool for solving a large class of differential equations, both linear and nonlinear. Many real-world applications have attested to the discrete wavelet transform's efficacy as a signal analysis tool. With its roots in 1910, Haar wavelets have shown to be incredibly useful in solving issues in science and engineering, and this paper aims to demonstrate just that. In addition, we cover the future of improving the Haar wavelet technique for solving differential equations, including its scope and directions.

Phang, Chang & Piau, Phang (2008) a relatively new mathematical method used for numerous issues is wavelet analysis, or the application of wavelet transformations. In numerical analysis, wavelets can be useful as well. Using known beginning or boundary conditions, we solve ordinary differential equations using Haar wavelet techniques in this study. We suggest a straightforward method to compute the matrix representation in order to facilitate the study of wavelets by novices and to spare them the pain of laborious calculations. In order to solve the highest-order differential, the method presented in this study requires integrating the Haar series. The four numerical examples span distinct types of differential equations: first-, second-, and higher-order, with constant and variable coefficients, respectively. When contrasted with strategies that seek a precise answer, the outcomes demonstrate that the suggested approaches are extremely realistic.

III. Haar Wavelet Matrices

In 1909, Alfred Haar introduced the Haar wavelet transform, which is the earliest wavelet that is known to exist. A haar wavelet is a square wave system with a scaling function on the initial curve, h_0.

$$h_0(x) = \begin{cases} 1 \; if \; 0 \leq x \leq 1 \\ 0 \quad otherwise \end{cases} \qquad (1)$$

and second curve is h1 is given by

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$$h_1(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2} \\ -1, & \frac{1}{2} \le x < 1 \\ 0 & otherwise \end{cases} \quad (2)$$

Mother wavelet is another name for this. Haar wavelets employ translation and dilation to execute the wavelet transform.

$$h_n(x) = h_1(2^j x - k),\ n = 2^j + k, \qquad j \ge 0, 0 \le k < 2^j \quad (3)$$

In addition, the haar functions may be used to approximate any square integrals function y(x) as

$$y(x) = \sum_{i=1}^{2m} a_i h_i(x) \quad (4)$$

Where $h_n(x) = [h_0(x), h_1(x), , , , , h_{m-1}(x)]$. Using the generic notation, we were able to represent the haar matrix as

$H_m = [h_m(1/2m), h_m(3/2m), , , , , h_m(2m - 1/2m)]$.

Thus we have

$$H_1 = (1), H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (5)$$

$$H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad (6)$$

$$H_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad (7)$$

Integration over the vector hm is given by

$$h_m = \int_0^x h_m(t)dt \approx P_m h_m(x), x \in [0,1] \quad (8)$$

Operational matrix Pm obtains the values as

$$P_1 = \left(\frac{1}{2}\right) \quad (9)$$

$$P_2 = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

$$P_4 = \frac{1}{16}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \tag{10}$$

$$P_3 = \begin{pmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{11}$$

IV. Numerical Method

Here we provide an example that might be used to implement the solution method discussed before. Here, we focus on the family of Haar wavelets as our basis functions. We answer the following problem in the following example:

$y'' + y = ex$

With $y(0) = 0$, $y'(0) = 0$ exact solution for this case is given by

$$y(x) = 1/2(ex - \cos x - \sin x) \tag{12}$$

To start the solution procedure, we have

$$y(x) = \sum_{n=1}^{2m} a_i h_i(x) \tag{13}$$

To find the values of $y'(x)$, we first had to integrate both sides and set $y'(0) = 0$. Then, we had to do it again by integrating the side and setting the value, and this time we got the

$$y(x) = \sum_{i=1}^{2m} a_1 p_{2,i}(x)$$

after putting the values of $y'' + y = ex$ in

$$\sum_{i=1}^{2m} a_i h_i(x) + \sum_{i=1}^{2m} a_i p_{2,i}(x) = e^x$$

The solution was obtained by simplifying this equation, as we discovered.

$$\sum_{i=1}^{2m} a_i [h_i(x) + p_{2,i}(x)] = e^x$$

After determining and entering the values of ai, we obtain $y(x)$. The numerical solution is obtained by substituting the values of $y''(x) = y(x) = ex$. The absolute error is then determined by comparing this solution with the actual one.

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Table 1: Comparative Results of Exact and Haar Solution

x	Exact Solution	Haar Solution	Absolute Error
0.125	0.0081	0.0088	0.0006
0.375	0.0791	0.0810	0.0019
0.625	0.2361	0.2399	0.0031
0.875	0.4952	0.4992	0.0041

If we take more collocation points and we will get more accurate results.

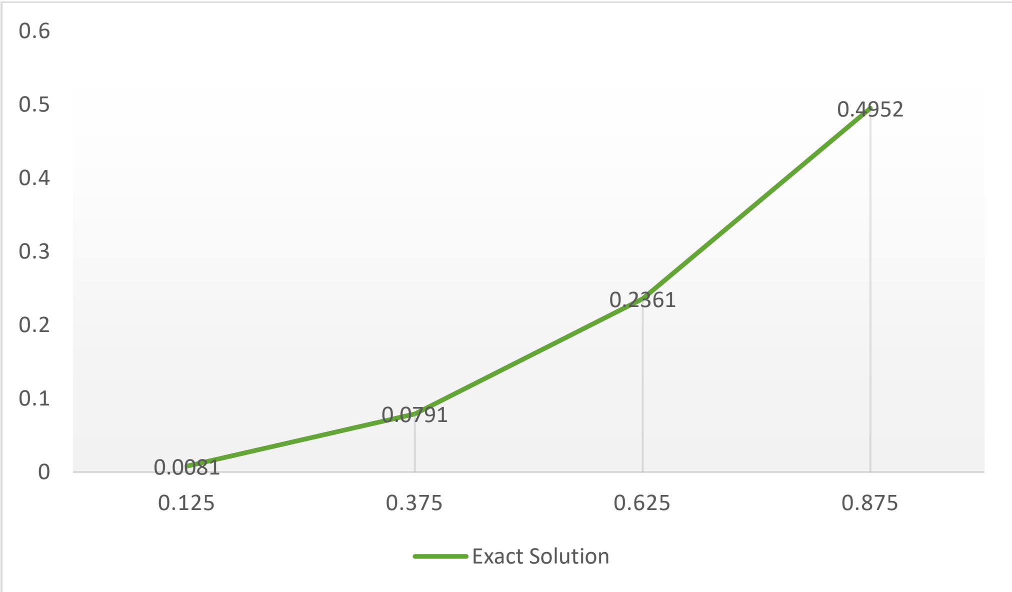

Figure 1: Results of Exact Solution

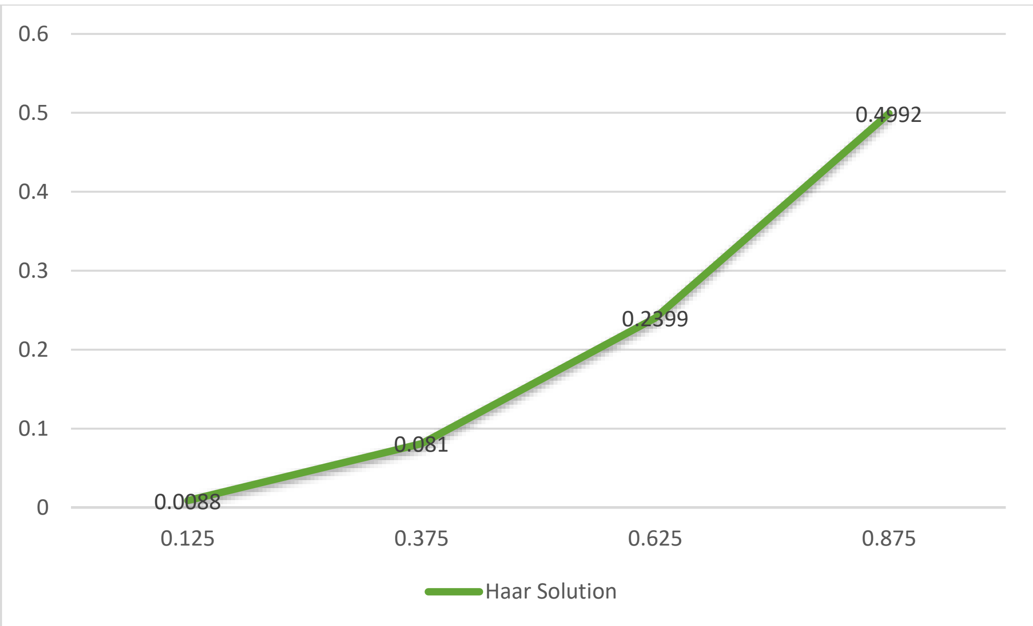

Figure 2: Results of Haar Solution

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V. Conclusion

By utilizing the operational matrix of integration, the differential equation is reduced to a system of algebraic equations, simplifying the computational procedure. The numerical solution obtained using Haar wavelet basis functions shows close agreement with the exact analytical solution, as evidenced by the small absolute errors at different values of xxx. The results clearly indicate that the accuracy of the solution improves with an increase in the number of collocation points. Due to its simplicity, low computational cost, and satisfactory accuracy, the Haar wavelet method proves to be an efficient numerical technique for solving ordinary differential equations and can be extended to more complex linear and nonlinear problems in applied mathematics and engineering.

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